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Understanding and prediction of turbulent flow—1996

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The title of this paper is also that of a lecture and a journal paper written by the present author 25 years ago (Bradshaw 1972). Note that "understanding" must come before "prediction"—although the order is reversed in the discussion below—because we must know what level of understanding is needed by currently affordable prediction methods. The present paper considers the question, "What progress has the human race made in the subject in the last quarter-century?" As always on these occasions, progress in some directions seems to have been slow, while other branches concerning the subject have advanced—or even come into being—with amazing rapidity. This paper deals mainly with momentum transfer, because (1) we must calculate the velocity field before being able to predict heat or contaminant transfer; and (2) momentum and thermal internal energy are transported by turbulence in roughly the same way.

Keywords: momentum transfer; turbulence; thermal internal energy

Introduction

Basic physics of turbulence

Turbulence is the most complicated form of fluid motion, and the most common. Research scientists can treat it as a fascinating phenomenon, while engineers regard it sometimes as an enemy, sometimes as a friend. The basic mechanism of turbulence is "vortex stretching." Turbulence is essentially three-dimensional (3-D), containing all three components of vorticity: as with any vorticity field, it can be thought of as composed of distributed vortex lines analogous to magnetic lines of force. It is deterministic, not completely random, but the motion is so complicated that the tools used in the study of random processes are useful in study turbulence. The interaction between the three components of vorticity leads to concentration of most of the vorticity into vortex tubes (the finite-thickness version of the vortex lines mentioned above, effectively real viscous vortices) and vortex slabs, which are the finite-thickness version of vortex sheets (surfaces occupied by nearly parallel vortex lines): Kelvin–Helmholtz instability, such as that found in transition from laminar to turbulent flow in a mixing layer, tends to make slabs roll up into vortex tubes connected by thinner slabs. (Transition and its prediction are not discussed in detail here; but note that breakdown to turbulence requires a 3-D perturbation, typically of a two-dimensional (2-D) instability mode such as Kelvin–Helmholz. Thus, the distributed vortex-line model is actually a usable approximation to the truth (e.g., Vincent and Meneguzzi 1994), rather than merely a mathematical abstraction to represent a smoothly varying vorticity field. The induced velocity field of all the vortex tubes leads, on the average, to the stretching of individual tubes, reducing their diameters: then, conservation of angular momentum implies that, as the diameter decreases, the rate of rotation (the vorticity) and the rotational kinetic energy both increase; that is, vortex stretching transfers kinetic energy from large wavelengths (vortex tube diameters) to small wavelengths—the "energy cascade." The start of the process is most turbulent flows is distortion of the larger-scale motion by the mean rate of strain (e.g., the mean shear in a shear layer), which transfers kinetic energy from the mean flow to the turbulence. The largest eddies have wavelengths on the order of the shear-layer width; e.g., the thickness of a boundary layer or the radius of a pipe. Most of the turbulent transport (mixing of mass, momentum, etc.) is not small-scale mixing, which might reasonably be modeled by a gradient-diffusion process, but bulk convection by the larger eddies, very roughly, those whose wavelength is more than a tenth of the shear-layer width or a tenth of the distance from the surface, whichever is smaller. The end of the process is dissipation of the kinetic energy of the smallest eddies into thermal internal energy by the action of fluctuating viscous stresses. The size of the smallest eddies decreases as the viscosity decreases (strictly, as the Reynolds number increases). The rate of kinetic energy transfer down the cascade to the smallest eddies is set by the large eddies, because they control the rate at which smaller eddies are stretched. Thus, small eddies do not have much effect on large ones, which considerably simplifies our problems in several respects. In particular, turbulent mixing is almost independent of viscosity, except close to a solid surface—the "viscous wall region" where all the eddies are "small." Furthermore, mixing is unaffected even by departures of the fluid constitutive relation from the simple Newtonian viscous-stress law, provided that non-Newtonian effects are confined to the smallest scales, and this makes the development of subgrid-scale models for large-eddy simulations much less critical (see the Large-eddy simulation section). The smaller the viscosity (strictly, the larger the Reynolds number)

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Received 10 March 1996; accepted 15 October 1996

Int. J. Heat and Fluid Flow 18: 45–54, 1997

655 Avenue of the Americas, New York, NY 10010

0142-727X/97/$17.00
PII S0142-727X(96)00134-8

the smaller the dissipating eddies are compared to the large, energy-containing eddies. That is, turbulence is not only 3-D and time-dependent but covers a large and continuous range of wavelengths.

Direct numerical simulation

The correct way to treat the "turbulence problem" is, of course, to solve the 3-D time-dependent Navier–Stokes (N–S) equations, which are almost universally believed to give an adequately exact description of turbulence in simple fluids. Today, we call this direct numerical simulation (DNS). The numerical problem, as such, is not too severe, but a finite-difference domain must cover the full width of the turbulent region (e.g., the thickness of a boundary layer) with a greater extent in the other two coordinate directions, while the mesh size must be small enough to resolve the smallest eddies. Millions of finite-difference points or spectral modes are needed even at modest laboratory Reynolds numbers, and calculations for such large engineering structures as aircraft or turbines are impossibly expensive at present. Therefore, DNS is at present only a basic research tool, supplementing—and beginning to compete with—experiments. However, any other approach to the turbulence problem is inexact; that is to say, a cheap answer to expensive problems.

Large-eddy simulation

Direct numerical simulation was a pipe dream 25 years ago, but meterologists had already begun what we now call "large-eddy simulation" (LES, Deardorff 1971). Here, we take advantage of the fact that most of the transport of momentum or thermal energy is carried out by the larger eddies. The small eddies are little affected by the boundary conditions of the flow, or, as Stewart (1969) put it in his excellent movie, "they don't know which way is up." In scientific language, they are very nearly isotropic, statistically speaking. Also, they are weak and contribute little to turbulent transport. This is why viscosity has little effect on turbulent mixing away from solid boundaries. We may, therefore, hope that an LES numerical calculation scheme, in which the large eddies are resolved by the finite-difference grid, and the small, statistically simpler subgrid-scale (SGS) eddies are modeled fairly simply, will give predictions almost as good as DNS, but at much less computing cost. This hope is being realised, although slowly. Large-eddy simulation is a rather crude simplification of the problem: cutting off the spectrum at an arbitrary wavelength (related to the finite-difference grid scale), where we change from full resolution to modeling, is likely to leave part of an eddy in the resolved motion and part in the SGS region, because a turbulent eddy is not a simple Fourier mode. (There is no exact definition of an "eddy": think of it as a region of correlated motion, as seen in say, a flow-visualization movie, and note that large eddies can contain smaller ones.) Another difficulty is that, as mentioned above, *all* the eddies near a solid surface are small, and they are also strongly anisotropic. Thus, either the SGS model must be good enough to deal with the full range of eddy sizes near the surface, or the grid size must be greatly reduced. The latter choice increases computer storage and calculation time, thus destroying much of the advantage of LES over DNS.

The first and simplest model for the SGS motion was that of Smagorinsky (1963). He assumed that the apparent turbulent stresses in the SGS motion are proportional to the square of the rate of strain in the resolved motion: his model is usually referred to as an eddy-viscosity model (the concept of eddy-viscosity is discussed below), but it is more akin to a mixing-length model. Note that most of the contributions to fluctuating velocity gradients, vorticity, and rate of strain come from the small eddies, so that the rate of strain in the resolved motion is determined mainly by eddies with wavelengths near the cutoff, which are likely to control the SGS motion. Now, DNS results can be used to test SGS models (admittedly only at low bulk Reynolds number); for example, fine-grid DNS data can be used to evaluate the constant of proportionality in Smagorinsky's model, or equivalently, the SGS eddy-viscosity, by calculating the actual SGS stresses and resolved-motion strain rate for a typical (coarse) LES grid. This produces values that vary wildy in space and time, but taking a time average, say, produces a smooth, and often fairly small, spatial variation. Unfortunately, different values of the "constant" are needed in different flows, and the lack of improvements to the Smagorinsky model led to a decline in research interest in LES in the 1980s in favor of DNS. Fortunately, new models have begun to appear.

The fluctuations in the DNS evaluations of SGS eddy-viscosity at, say, a given point in space are large enough to produce negative values at some times. This indicates—independently of the details of the SGS model—that the SGS motion is not dissipating energy locally but feeding it back to the resolved motion. This phenomenon is called "backscatter": it is a consequence of the fact that vortex lines (see the Basic physics of turbulence section) are not stretched monotonically but sometimes contract.

The most promising of the recent SGS models is the dynamic model of Germano, in which the eddy-viscosity of the SGS motion is related to the apparent eddy-viscosity of the smallest resolved eddies: in the original form of the model, the appearance of negative eddy-viscosity leads to numerical instability, but Ghosal et al. (1995) describe the most recent version, in which a transport equation is resolved for the SGS kinetic energy and used to cut off the backscatter at a given point when the SGS kinetic energy at that point falls to zero.

Reynolds-averaged models

Even LES currently costs more than engineering companies are willing to pay for routine calculations, and, of course, engineers do not want a complete output of all three instantaneous velocity components and the instantaneous pressure as functions of x, y, z, and t: they only want the statistics, and usually only the very basic statistics, at that. This brings us to the really sweeping simplification of Reynolds averaging. We suppose for simplicity that the average is a time average or "mean": this is a special case, but by far the most common one. Information is lost by averaging and must be replaced by one or more equations with empirical coefficients. "Leading" coefficients are the basic ones, such as the eddy-viscosity coefficient c_μ in the two-equation k–ε model: the adjective is needed, because the leading coefficients are sometimes made functions of, e.g., the ratio of the rate of production of turbulent kinetic energy to its rate of dissipation, and, of course, the subsidiary coefficients of such a function must be constant for the function to be incorporated into a computer code to be run without human intervention. Reynolds averaging throws away so much information that it is generally, although not universally, accepted that no model based on Reynolds-averaged equations can ever give predictions to *acceptable engineering accuracy*, over the full range of flows of interest, with constant values for the leading empirical coefficients in the model. The empirical coefficients must be obtained from experimental data or simulation results, either by direct measurement or by trial-and-error adjustment to optimise the model predictions for the flows of interest. A recent technique that seems to avoid the constraint in the last sentence is the use of renormalization group theory (RNG: see, e.g., Nakamur and Sakya 1995, who discuss the use of RNG-based models to predict transition). Renormalization group theory analysis nominally deduces the behavior of the large eddies from that of the smaller ones using the scale-similarity inherent in the energy cascade (see the Basic

physics of the turbulence), but scale similarity does not extend to the range of (large) eddy size that carries the Reynolds stresses. The RNG analysis involves choosing a "small" parameter to be larger than unity (equal to 4 in one version at least), which seems questionable. Renormalization group theory produces the right form of the terms in, say, one of the two-equation models, but this is essentially dimensional analysis, and it is not clear where the proper physics enters the process. However, good results have been obtained.

Note that by "acceptable engineering accuracy" we mean "accurate enough for design and/or performance prediction of aircraft, turbines, and other critical engineering devices" implying errors of not more than a very few percentage points in such quantities as skin friction and shear-layer growth rate. Less accurate predictions may still be a useful guides, provided that the error can be estimated; however, in the present state of modeling, unexpectedly large errors can show up in flows different from those for which the model has been calibrated.

Reynolds averaging, as is well known, results in equations for the mean velocity and temperature fields that are the same as in steady laminar flow, with the addition of gradients of extra apparent stresses or heat fluxes representing the extra transport of momentum or thermal energy by the turbulence. The label "stress" for what is really a rate of momentum transfer is an inexact one, but, of course, its use is no more inexact in turbulence than in the kinetic theory of gases, where momentum transfer by molecular collisions corresponds, in the macroscopic view, to what we call viscous stresses. Further manipulation of the N–S transport equations for (instantaneous) momentum leads to transport equations for the Reynolds stresses, shown in Equation A1 (they are partial differential equations whose left-hand sides are the time-averaged substantial derivatives of the components of the Reynolds stress tensor, just as the left-hand sides of the three N–S equations are the instantaneous substantial derivatives of the components of instantaneous velocity or momentum). The right-hand sides contain further unknown statistical quantities, but the first group of terms contains no new quantities: this is the so-called generation of Reynolds stress by the effect of the mean-flow distortion on the existing turbulence and is a sum of products of Reynolds stresses and mean velocity gradients. The form of the generation terms shows that the mean velocity gradients influence the rate of change of Reynolds stress along a mean streamline but do not set its local value. This means that local-equilibrium concepts taken over from such kinetic theory of gases as the concept of eddy-viscosity are very restricted in their application to turbulence.

In almost all parts of a turbulent flow, turbulent (Reynolds) stresses or heat fluxes are far larger than the viscous/conductive transport rates, the exception being the above-mentioned "viscous wall region" very close to a solid surface. Velocity fluctuations and, thus, turbulent stresses are zero at the surface because of the no-slip/no-permeability conditions. As a related but slightly different point, the turbulent stresses are nearly independent of viscosity (recall that the smallest, viscous-dependent eddies do not affect the larger, stress-carrying ones). This independence of viscosity and conductivity greatly simplifies the task of modeling the stresses and heat flux rates.

"Modeling," in the present context, means replacing the higher-order unknowns in Reynolds-averaged equations with functions of the dependent variables of the equations, so as to achieve a closed set of equations (number of unknowns equal to number of equations). Details vary greatly: one constraint that must be obeyed is that mean velocities should appear in the modeled terms only as gradients, because the appearance of the velocities themselves would make the model depend upon the frame of reference. Equations for turbulent stresses or flux rates are then solved together with equations for the mean velocity field. The last sentence immediately suggests the hope of a *direct* connection between the Reynolds stresses and the mean velocity (necessarily the first or higher spatial derivative of the mean velocity, for the sake of Galilean invariance) and/or between the turbulent heat-flux rate and mean temperature gradient. If we suppose that the stress is proportional to the mean rate of strain, we have invented eddy-viscosity, and there is an analogous "eddy-conductivity." They are *not* properties of the fluid: they change, in a complicated fashion, from flow-to-flow and from place-to-place in a given flow. The ratio of a turbulent stress to the rate of strain in the same plane can always be defined, and, of course, measured, so that the eddy-viscosity ν_T defined by

$$-\overline{u_i u_j} = \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{1}{3}\delta_{ij}\overline{u_l^2} \tag{1}$$

is a respectable quantity. It has the disadvantage of going to infinity, if the rate of strain is zero, and the stress is not; and, less obviously, there is no reason why it should be a scalar, the same for all choices of i and j. Almost all turbulence models that use an eddy-viscosity assume that it *is* a scalar ("isotropic"), and, indeed, it is difficult to define an anisotropic eddy-viscosity without relating it to special directions defined by the boundary conditions (e.g., "spanwise" and "streamwise" components), which is usually incorrect, because it violates the principle of rotational invariance (changing the axes should not change the equations).

It is important to note that eddy-viscosity is the ratio of a turbulence quantity to a mean flow quantity, and we should not expect it to follow the scales of the mean flow exclusively, nor those of the turbulence. Nevertheless, current eddy-viscosity models fall into two main types. The first type includes the so-called "algebraic" models that relate the eddy-viscosity to mean-flow parameters, a classic example being the correlation $\nu_T = 0.0168 U_e \delta^*$ for the outer region of a boundary layer (Cebeci and Smith 1974). Algebraic eddy-viscosity models are a simple and cheap choice for not too demanding flows and can often outperform more sophisticated models in boundary layers in pressure gradient. The second type of eddy-viscosity model includes "two-equation" models, in which partial differential equations are solved for two statistical-average scales of the *turbulent* motion. The absence of mean-flow scales means that such models can be applied to any geometry (accuracy, of course, is not guaranteed).

The two-turbulence scales of a two-equation model are a velocity scale (almost always the square root of the turbulent kinetic energy per unit mass $k \equiv \overline{u_i^2}/2$) and another quantity yielding a length scale of the energy-containing eddies; the eddy-viscosity is then taken as some numerical coefficient times the product of the two scales. Note that *all* models, whether of the eddy-viscosity type or not, need a length scale or time scale as well as a velocity scale, because the terms in the transport equation for a variable Q, say, have dimensions of Q/time or $Q \times$ (velocity/length). Of course, a time scale can be combined with a velocity scale to give a length scale, and it is usually simpler to discuss the latter. The most popular two-equation model is the k–ε model attributable to Jones and Launder (1972; see Equations A3 and A4), in which ε, the rate of dissipation of turbulent kinetic energy per unit mass k, yields a length scale $k^{3/2}/\varepsilon$. The eddy-viscosity ν_T defined by Equation 1 is then given by $c_\mu k^2/\varepsilon$ where c_μ is an empirical coefficient. (Note that the definitions of typical length and velocity scales need not involve numerical factors: the latter appear in the modeled terms, as in the case of c_μ, which is typically taken as 0.09). An exact transport equation for ε can be derived from the

Navier–Stokes equations, but it contains many complex terms, and the dissipation equations used in current models must be regarded as completely empirical.

A third type of eddy-viscosity model is the "one-equation" model, which solves a partial derivative equation (PDE) for ν_T directly. It still needs a length scale. The first model of this type was that of Nee and Kovasznay (1968) but it contained the shear-layer thickness as a length scale and thus was not independent of flow geometry. Interest in one-equation models has revived recently, and the model of Spalart and Allmaras (1994), in particular, seems to be at least as good as two-equation models in some flows.

In principle, a better method of predicting Reynolds stresses, which does not rely on the eddy-viscosity concept, is the "stress-transport model" or "second-moment closure." Here the exact Reynolds stress transport equations are modeled term by term. The result is a set of PDEs for the Reynolds stresses, to be solved with the Reynolds-averaged momentum equations and continuity equation. Either the exact equations or the models show that the mean rate of strain (strictly, the various mean velocity gradients) helps to determine the *rate of change* of each Reynolds stress, but not directly the magnitude of the stress. It is now clear why eddy-viscosity is a bad variable to use for empirical correlations: it assumes a *direct* connection between stress and rate of strain. A major term to be modeled in each stress equation is the mean product of the pressure fluctuation with the rate-of-strain fluctuation in the same place as the stress: this "pressure-strain" or "redistribution" term cannot be measured reliably, and data from simulations are limited, but as this *is* a major term, it can be deduced reasonably accurately as the net sum of all other terms in the stress equation (e.g., Schwarz and Bradshaw 1994). Stress-transport models still need a length scale, usually supplied by the same kind of PDE as that used in two-equation eddy-viscosity models. As in two-equation models, the most popular choice is a PDE for ε.

Stress-transport models have not fulfilled their promise of 25 years ago (e.g., Hanjalic and Launder (1972)) despite a considerable amount of refinement over this period, and their superiority over two-equation models is sometimes significant, sometimes not: opinions are divided as to whether the fault is mainly that of the dissipation equation or of the pressure-strain model. Stress-transport models take longer to run than two-equation models, but numerical reliability of both has been improved over the last 10 years or so by reformulating models to ensure "realizability" (i.e., no negative mean-square intensities or dissipation rates). Another requirement that some models have been modified to satisfy is correct limiting behavior at a solid surface, where the turbulence is strongly affected by viscosity; however, it is a matter of opinion whether this should constrain the whole model, as used in the main part of the flow where viscous effects on the Reynolds stresses are negligible, or simply be incorporated in the "low-Reynolds number" version used in the viscous wall region.

"Algebraic stress models" are a simplification of stress-transport models and emerge as two-equation models with an anisotropic eddy-viscosity. The assumptions needed are sweeping. In particular, it is assumed that the *left*-hand sides of the stress-transport equations (i.e., the "mean transport terms") are proportional to the stress being transported, so that all can be obtained from a solution of, say, the turbulent kinetic energy equation, in the following form:

$$\frac{D\overline{u_i u_i}}{Dt} = \left(\frac{\overline{u_i u_j}}{k}\right)\frac{Dk}{Dt} \tag{1.2}$$

It is easy to show that this assumption is poor in flows where the mean strain rate changes rapidly, and, of course, this is just the kind of flow where we need something better than a straight eddy-viscosity model. A similar assumption is used to relate the *turbulent* transport of each Reynolds stress to the turbulent transport ("diffusion") and Fu et al. (1988) have shown that this, too, is unsatisfactory where turbulent transport is large—for example, near the edge of a free shear layer. It now seems that algebraic stress models are not a substantial improvement over two-equation models.

A basic question in calibrating models of any level of complexity is the range of flows over which the model can be expected to be reliable. For example, one of the key coefficients in the transport equation for dissipation (or other "length-scale" quantity) can be obtained from the experimentally obtained law of decay of isotropic turbulence. To do this and then apply the model to shear layers implies the assumption that the key coefficient is actually the same in isotropic turbulence and in shear layers. This is likely to be an inaccurate assumption, because properties such as spectrum shape and higher-order moments of the fluctuations are very different in the two types of flow.

As has been implied in several places above, another basic question is how to treat the boundary condition at a solid surface, or more specifically, how to treat the viscous wall region. In this region, the coefficients in the model can in principle—and generally do in practice—become functions of a Reynolds number based on the local velocity and length scales of the turbulence, adding further empiricism to the model. Over a wide range of attached flows, the near-wall region is nearly universal: in the case of the mean velocity this is the well-known "law of the wall." This justifies making the coefficients functions of the local turbulence Reynolds number, independent of the type of flow. However, evidence is accumulating (e.g., Bradshaw and Huang 1995) that law-of-the-wall arguments can break down quite seriously in flows that we would not normally think of as complex. In particular, the range of validity of the law of the wall for temperature is very small—it fails spectacularly in boundary layers in pressure gradient—which is disquieting, because the arguments used to derive the law of the wall for temperature are closely analogous to those used for velocity. This is a serious matter, because model predictions for wall flows depend strongly on the accuracy of treatment of the viscous wall region. Launder and Tselepidakis (1991) de-emphasise the role of the turbulence Reynolds number, and it seems likely that the main effect of the wall may come from the no-permeability condition rather than the no-slip (viscous) condition.

Two-equation or stress-transport models become very stiff, numerically, near the wall. The stiffness can be reduced by specifying the length scale and integrating a PDE for the velocity scale k only ("one-equation modeling," see Rodi et al. 1993). There seems to be no detectable reduction in accuracy.

It is possible to avoid introducing Reynolds number-dependent coefficients by satisfying the "surface" boundary condition some distance outside the viscous wall region but still within the range of validity of the law of the wall. However, this still depends on near-wall universality, is probably accurate over a smaller range of flows than integrating down to the wall, and is complicated numerically.

The various types of turbulence models are discussed in more detail in two recent references (NASA 1995; Bradshaw 1996).

Types of turbulent flow

Because turbulence is a dissipative phenomenon, most flows that survive long enough to be of interest to engineers or planetary scientists have an energy input to the turbulent motion. One example is production of turbulent kinetic energy by buoyancy effects, at the rate $-\overline{g\rho' v}/\rho$ per unit mass, where v is the velocity fluctuation in the upward vertical y direction, and ρ and ρ' are the mean and fluctuating parts of the density. Negative values of $\overline{\rho' v}$ (transfer of potential energy *to* the turbulence) are

usually associated with positive $\partial\rho/\partial y$ (heavy fluid above light fluid—an unstable situation) and conversely. An obvious case is the atmosphere, where warming of the air near ground heated by the sun leads to instability and the generation of "thermal" updrafts and corresponding downdrafts. Buoyant convection can take place without a mean flow in the horizontal direction.

The most common kind of turbulent flow in engineering is one dominated by one or more *shear layers* (the family name for boundary layers, wakes, jets, etc.). Supposing that the mean flow is in or near the x-direction and is independent of the lateral coordinate z (the special case of a "two-dimensional thin shear layer"), the rate of transfer of kinetic energy per unit mass from the mean flow to the turbulence is $-\overline{uv}\partial U/\partial y$. Note that $-\rho\overline{uv}$ is the shear stress in the x–y plane so that the energy transfer rate per unit volume is (*stress*) × (*rate of strain*), which is a rate of doing work (a factor of 2, resulting from tensor summation, has been glossed over here). In real life, shear layers are often 3-D (dependent on z as well as x and y), curved in the x–y or y–z plane, or affected by adjacent turbulence fields, body forces, passage through a shock wave, and other imposed perturbations. Nevertheless, these perturbed or "complex" shear layers usually dominate the flow: they are an ongoing subject for experiments and an ongoing difficulty for prediction methods. If the perturbation can be described simply (e.g., by the radius of a curved shear layer), extra empirical terms can be added to a prediction method to account for it: many modelers feel that this is undesirable and that, to be reliable, a model should reproduce complex-flow effects naturally.

However, even the most advanced models seem unable to predict the effects of streamline curvature in the x–y plane (an extra mean-velocity gradient $\partial V/\partial x$) without extra curvature-dependent terms, and the same seems to be true of other kinds of perturbation in the form of additional mean-velocity gradients. The direct numerical simulations of Coleman et al. (1996) show that even $\partial U/\partial x$; i.e., longitudinal pressure gradient, significantly affects turbulence structure. Mean pressure gradients do not directly affect turbulence, at least in incompressible flow, and the usual view (expressed, e.g., by Bradshaw 1994) is that there is little evidence for indirect effects via $\partial U/\partial x$. However, the effects are somewhat difficult to see in experiments on flows in pressure gradients because of large streamwise rates of change in the flow as a whole: this is a good example of the value of idealised simulations.

A very common perturbation imposed on a shear layer is the addition of three-dimensionality to the mean flow (Bradshaw 1987). A typical example is an initially 2-D boundary layer encountering a pressure gradient in the lateral z direction, leading to nonzero W and nonzero $\partial W/\partial y$ (even in inviscid flow, lateral skewing of the mean velocity vector skews the vorticity vector away from the z-axis so the x-component vorticity $\partial W/\partial y - \partial V/\partial z$ becomes nonzero). Conventional turbulence models do a poor job of predicting the development of the Reynolds stresses even in simple 3-D flows and do even worse in "slender" flows with strong three-dimensionality in a limited part of the cross section, such as flows in wing-body junctions or duct corners (the latter problem is discussed by Speziale et al. 1993). In summary, although most of the flows of interest are identifiable as members of the family of shear layers, they are usually significantly perturbed from the "textbook" shear layers, in a way that current turbulence models cannot predict reliably.

Experiments

Direct numerical simulation needs no empirical input (other than the principles of conservation of mass and momentum, and the viscous stress law, which combine to give the Navier–Stokes equations). Experimental data are not of much use in formulating and testing SGS models for LES: only DNS results give the detail required. The main role of experiments is the traditional one of providing information for developing and testing Reynolds-averaged models: we use the word "information" rather than data, because even an experiment that reveals only qualitative physics can help in the general formulation of a model, leaving the empirical constants to be determined from quantitative data. In recent years, the results of simulations, both DNS and LES, have become useful supplements to experimental data, although simulations are restricted to fairly simple geometries and low Reynolds numbers and are, therefore, more helpful in formulating a model than in providing general test cases. In the last 25 years, the appearance of laboratory computers for controlling an experiment and recording the data, and a number of new techniques mainly based on laser light sources, have greatly increased the range of measurements and the complexity of the flows that can be measured.

It is difficult to review the advances in complex-flow measurements, but the reader may look at the description of any recent experiment and convince him or herself of the virtual impossibility of acquiring such large volumes of data, and deriving complicated statistics, by manual control and analog processing. In the first 10 years or so of our quarter-century, computers were expensive facilities, centralized in the laboratory or the organization rather than dedicated to a single experiment, and digital data processing often required recording on analog magnetic tape with later transcription to digital storage. Today, inexpensive microcomputers are fast enough for most data-acquisition purposes, and data transfer to larger machines with, say, more advanced graphics facilities is straightforward. (A student of the author's recently gave a seminar on measurements in a 3-D complex flow, using the NASA Ames FAST graphics package to produce color contour plots of surface pressure and other variables from data on a fine grid. An inattentive member of the audience asked if these were experiments or computations!)

A notable development that began about 25 years ago (Kovasznay et al. 1969) is "conditional sampling," the accumulation of statistical averages only over those periods of time for which the flow satisfies some condition chosen by the experimenter. A simple example is an "intermittent average" near the irregular free-stream edge of a turbulent flow, which is accumulated only over the periods for which the flow at the measurement point is turbulent. This can be done quite easily using either analog or (more commonly today) digital data processing, although deciding on a criterion for "turbulent" can be nontrivial: strictly speaking, we should measure the magnitude of the fluctuating vorticity and label the flow turbulent when it is nonzero (not counting zero crossings); however, in practice, less rigorous criteria are used. If the Prandtl number of the fluid is close enough to unity, heat can be introduced into the turbulent flow, and the edge of the heated fluid (detectable with a fast-response thermocouple or resistance thermometer) will coincide with the edge of the turbulent fluid. Introduction of heat to mark only part of the turbulent flow, e.g., one of a pair of merging shear layers or a shear layer embedded in free-stream turbulence, can also be informative (e.g., Hancock and Bradshaw 1989).

Conditional sampling of simulation data is somewhat easier than experimental work, because complete information about the flow is available. Perhaps the most impressive example of this is the work of Robinson (1991). An important example in the present context of heat transfer is the little-known work of Guezennec et al. (1990) and Stretch et al. (1990) whose analysis of DNS results for a heated channel flow showed that the ejections or "bursts" of heated fluid from the viscous wall region were much more slender than the ejections seen in the velocity field. The reason is simply that pressure fluctuations affect the velocity field, but in a low-speed flow—not the temperature field.

Presumably the ejections of vorticity (also unaffected by pressure fluctuations) resemble the hot-fluid ejection rather than the velocity patterns. These differences between the mechanisms of heat transfer and momentum transfer make it very surprising that the turbulent Prandtl number (ratio of turbulent momentum diffusivity to turbulent heat diffusivity) is close to unity, about 0.9, in the inner layer of a simple wall flow. Reynolds analogy does not bear inspecting too closely! Note that the *constancy* of turbulent Prandtl number in the inner layer follows from law-of-the-wall arguments, which should apply to the temperature field as well as the velocity field, although, in fact, they seem to be valid only in a very simple flows (Kays 1994; Bradshaw and Huang 1995). However, these arguments do not imply that the constant value of turbulent Prandtl number must be near unity.

A great deal of attention has been devoted, in the last 20 years or so, to coherent structures. These are eddies that have an unusually long lifetime, and so the class depends on our concept of "unusual." The paper that created the wave of interest was that of Brown and Roshko (1974), showing the persistence of quasi-two-dimensional structures (vortex rolls) in mixing layers. After a search for similarly coherent structures in other kinds of flow, it became clear that the mixing layer is a rather special case: the presence of a point of inflexion in the mean velocity profile allows the eigenmodes of Kelvin–Helmholtz instability to persist, even against a background of small-scale turbulence. Wakes (with two points of inflexion) apparently have less pronounced coherent structures, while the vortex *rings* that are the eigenmodes of an annular mixing layer break up into conventional 3-D turbulence as the mixing layer spreads to form a jet. More or less by definition, if an eddy persists for a time much longer than the natural time scale k–ε (i.e., the time that the existing rate of dissipation would take to reduce the existing turbulent kinetic energy to zero), then that eddy is not fully a part of the turbulent energy cascade (see the Basic physics of turbulence section). Although there is still some dispute, it appears that the cores of the vortex rolls in a mixing layer are in nearly solid-body rotation without much turbulent activity. The present consensus is that the orderly structures in flows other than mixing layers are simply the large eddies of characteristic statistical-average shape first discussed by Townsend (1956). The large eddies carry much of the turbulent kinetic energy and shear stress, and their lifetime is no more than a few times k–ε. There have been several attemps to represent turbulent flows with a combination of deterministic large eddies and background isotropic turbulence. It seems that this combination is too crude to yield results of engineering accuracy, but see Goldshtik and Hussain (1995) for an interesting development in the special case of mixing layers. The other contributions of coherent-structure research to the development of prediction methods have, sad to say, been few and disappointing.

Among the techniques for turbulence measurement that have come into widespread use over the last 25 years are the laser-Doppler velocimeter (LDV) and particle image velocimetry (PIV). There are variants of each. Useful references for LDV are the old but still valid reviews of Durst et al. (1976) and Buchhave et al. (1979) and the discussion of errors by Gould and Loseke (1993). Lourenco and Krothapalli (1995) discuss PIV errors.

The advantage of the LDV over, say, the hot wire is that no solid probe need be inserted into the flow. Effects of a probe on the flow, or of a hostile fluid on the probe, are eliminated. The disadvantage is the cost, certainly an order of magnitude more than a hot-wire system to do the same job. The simplest explanation of the LDV is that if two coherent beams of light intersect, then planar interference fringes will be formed, and the intensity of light reflected by a particle passing through the intersection volume will fluctuate. The fluctuation frequency is equal to the particle's velocity component normal to the planes of the fringes, divided by the fringe spacing. If the signal from a photomultiplier viewing the particle is put through a frequency-to-amplitude conversion system—the simplest example being the demodulator used in FM radios—a voltage proportional to the velocity component is obtained. "Seeding" the flow with particles is an art: it is not always possible to achieve a particle density large enough to give a continuous FM signal but not so large that the combined signal smears out the (random phase) fluctuations. Statistics that do not involve the time domain can be obtained from rare particle crossings, but, in this case, nonuniform or localized injection can lead to permanent correlation between particle number density and the flow near the injection point, thus biasing the results. A given beam pair can measure only the velocity component in the plane of the beams and normal to the axis of symmetry: to measure three components requires three beam pairs, of different colors so that the three signals can be distinguished.

The basic principle of PIV is to seed the flow completely with particles and then to store the image of two or more short-duration pulses of a light sheet intersecting the flow. The distance a particle image travels between the pulses is proportional to its velocity in the plane of the light sheet, so that two velocity components can be recovered from each particle in the plane, again without interfering with the flow. Usually the pulses are chosen to be of unequal length, -·, or unequal spacing ···, so that the sign of the velocity along the displacement vector can be deduced. Image processing is (computer) time-intensive, and care is needed to avoid spurious results caused by confusion between adjacent particles and particles that do not stay within the thickness of the light sheet for the full duration of the pulse train. Also, the results need to be interpolated on to a uniform grid for practical use, and, of course, many images are needed to get well-converged statistical averages. Useful review of the state of fluid measurement techniques as of 1988 are given in the books edited by Gad-el-Hak (1989a, b), especially the former.

The present state of Reynolds-averaged models

Wilcox (1993) contains a general review of turbulence models, concentrating mainly on two-equation models.

Algebraic eddy-viscosity "zero-equation" models (containing no PDEs), and the even older "integral" methods solving ordinary differential equations for momentum thickness, shape parameter, etc., are suitable only for simple attached boundary layers and have not been developed much in the last 25 years. An exception is the algebraic eddy-viscosity model of Baldwin and Lomax (1977), which is essentially the same as that of Cebeci and Smith (1974, section 1.4) with the displacement thickness replaced by a length scale that is well behaved even if the edge of the boundary layer is ill-defined because of nonzero $\partial U/\partial y$ in the external stream. However it is fair to say that if $\partial U/\partial y$ does not go substantially to zero at the edge of the boundary layer, then the boundary-layer approximation (which is implicit in the model equations) is suspect. The Baldwin–Lomax model is extremely robust numerically; i.e., seldom breaks down. An unfortunate consequence is that it has ben applied (not by its originators) to cases such as highly 3-D flows, which are far beyond the range of validity of the algebraic eddy-viscosity correlation and for which no adequate experimental data exist to check the predictions.

Of the one-equation models (PDE for eddy-viscosity), the Spalart–Allmaras model seems to be superior to that of Baldwin and Barth (1990). The latter was derived as a simplification of the two-equations k–ε model; whereas, the former was constructed empirically by a building-block approach, starting with simple cases and then adding complications. The building-block approach does rely on the empirical coefficients in the model

keeping the same optimum values for the full range of flows, but is evidently a way of making the best use of available data. Spalart and Allmaras (1994) refer to the one-equation model of Secundov and colleagues, which began as a refinement of the original empirical Nee–Kovasznay (1969) model, to avoid explicit use of the shear-layer thickness as a length scale, but is a precursor of Baldwin–Barth and Spalart–Allmaras. It seems that one-equation models will always be limited by the lack of an explicit length-scale equation. A geometry-independent model can be formulated by—in effect—obtains the length scale from the ratio of ν_T to the rate of strain, which has dimensions (length^2). Spalart and Allmaras' (1994) use of the mean vorticity, rather than the mean strain rate, to provide a time scale for the turbulence has the advantage that the time scale does not change rapidly when the streamline curvature in the x–y plane changes. As with most other models, Spalart and Allmaras' does not reproduce the suppression of turbulent mixing in the core of a longitudinal vortex (Dacles-Mariani et al. 1995) and, in general, seems to need an extra term to represent curvature effects.

The two-equation k–ε model was originally from Jones and Launder (1972), but the empirical constants in common use are those recommended by Launder and Sharma (1974). Model Equations A3 and A4 are solved for k and ε to give the eddy-viscosity.

$$\nu_T = c_\mu k^2/\varepsilon \tag{2}$$

It gives results that are of at least semi quantitative use to engineers over a wide range of flows. It is well behaved numerically except for stiffness in the near-wall region; this necessitates small x-steps if the integration is carried all the way to the wall rather than being matched to a "wall function" (logarithmic law or otherwise) somewhere outside the viscous wall region. k–ε is notoriously unsatisfactory for the apparently simple case of boundary layers in adverse pressure gradients, which even algebraic eddy-viscosity methods can handle quite well. As with most other models, k–ε is calibrated to reproduce the logarithmic law (strictly speaking, the mixing-length formula) in the inner layer of a wall flow, and the worst errors seem to occur just outside the inner layer. The modeling required in the k (turbulent kinetic energy) equation, A2, is minor, confined to the turbulent transport term (see Equation A3), so the main suspicion falls on the eddy-viscosity formula—with some doubts about the dissipation equation. S. F. Birch (private communication 1995) points out that the k–ε is tuned to deliver the best possible result for $c_\mu k^2/\varepsilon$ as required by the eddy-viscosity formula, so that comparisons between experiments and predictions of k and ε separately may be misleading; for example, a too-low choice for c_μ could be compensated by a too-low prediction of ε.

k–ε is a member of the general family of k, $k^m\varepsilon^n$ equations ($m = 0, n = 1$). The k, ω model ($m = -1, n = 1$) of Wilcox (1993) seems to give better results in many cases, including boundary layers in adverse pressure gradient and compressible flows but is excessively sensitive to the free-stream values of k and $\omega \propto \varepsilon/k$. Menter (1992) produced a hybrid model that used k and ω in the inner part of a wall flow, but k and ε in the outer part. Cazalbou et al. (1994) showed that the k–ε model happens to be rather insensitive to the choice of free-stream boundary values.

Apart from seeking the optimum m and n, the main mileage left in two-equation models lies in making the coefficients, notably c_μ in Equation 2, functions of dimensionless quantities parameterizing special effects. The most obvious "special effect" is that of viscosity, and the original Jones–Launder model incorporated low-Reynolds number modifications right from the start. The relevant Reynolds number is $k^2/(\varepsilon\nu)$, which is roughly proportional to y^+, and if the flow is fully turbulent, the modifications become active only in the viscous wall region. There is controversy about the ability of turbulence models, of any kind, to predict transition (where $k^2/(\varepsilon\nu)$ is also low). Results of at least qualitative value can be obtained for so-called "breakthrough" transition under highly turbulent streams, and for reverse transition ("relaminarization") but models calibrated in fully turbulent flow can hardly be expected to reproduce the growth of initially infinitesimal disturbances such as Tollmien–Schlichting waves.

Other special effects that have been allowed for by parameterizing c_μ include streamline curvature, or rotation, in the x–y plane. In the case of rotation about a spanwise axis with angular frequency Ω, the simplest relevant dimensionless parameter is $\Omega/(\varepsilon/k)$, replaced by $(U/R)/(\varepsilon/k)$ in the case of curvature, and empirical correlations for the effects of these parameters on c_μ have been quite successful. Stress-transport models (and algebraic stress models) have exact terms that include curvature and other mean-velocity gradients additional to the simple shear $\partial U/\partial y$, but the effect of these terms on the predicted stresses is generally too small by an order of magnitude. When longitudinal curvature changes, the turbulence structure is slow to respond, so correlating curvature effects on a local parameter may be unrealistic. A more refined approach is to modify the coefficients in the transport equations for k and/or ε, but there is no consensus on how to do this.

The prototype of present day transport-equation models is the thin shear-layer model of Hanjalic and Launder (1972), extended to general flows by Launder et al. (1975) and known as the LRR model. The exact stress-transport equations are A5, with the definitions A6 and A7, for up to six independent Reynolds stresses $-\rho\overline{u_iu_j}$. They are modeled term by term using a model dissipation transport equation (Equation A8), which is essentially the same as in the k–ε model. The major terms in the Reynolds-stress transport equation for $-\overline{u_iu_j}$ are the "generation" terms $-(\overline{u_iu_l}\partial U_j/\partial x_l + \overline{u_ju_l}\partial U_i/\partial x_l)$, and the pressure-strain "redistribution" terms $\overline{p'(\partial u_i/\partial x_j + \partial u_j/\partial x_i)}/\rho$, which usually have the opposite sign to the generation. Because the dependent variables in the model include the Reynolds stresses and mean velocity, the generation terms can be left in exact form, but the pressure-strain terms (and some turbulent-transport terms that are generally smaller) must be modeled.

Much of the effort expended on stress-transport models in the last 25 years has related to the pressure-strain terms. The exact Poisson equation that describes the pressure fluctuations in turbulent flow has a right-hand side consisting of two sorts of terms, one of which contains the *mean* velocity gradients. This surprising result, that a fluctuating quantity depends directly on a mean quantity, is just the result of Reynolds averaging and does not imply unexpected physics. However, it does imply that models for the pressure-strain term should also have two parts, the "rapid" part corresponding to the Poisson term containing mean velocity gradients, and the "slow" part depending only on Reynolds-averaged turbulence quantities. (The names are slightly confusing and arise from the response to an idealized case where the mean velocity gradient changes rapidly: the "rapid" part of the pressure-strain term changes at once, the "slow" part more slowly. In more conventional flows, the two parts respond at roughly the same rate.) In unsheared homogeneous flows, only the slow part of the pressure-strain term appears and controls the return of the turbulence to isotropy (shear-stress components zero, normal stresses equal). Early models assumed that the slow term was linearly proportional to the anisotropy; e.g., proportional to $-\overline{u_iu_j}$ if $-\overline{u_iu_j}$ is a shear stress, $i \neq j$. More recently, nonlinear models have been proposed (see the tests by Schwarz and Bradshaw 1994 and the model of Chung and Kim 1995). The "rapid" part of the pressure-strain term directly opposes the generation, and in the case of rapid distortion of initially isotropic turbulence (sudden application of a very large strain rate), an

exact result shows that the rapid pressure-strain term is just $(-3/5)$ times the generation term. Now if the empirical coefficient in the model of the rapid pressure-strain term is the same in rapid distortion and in conventional shear layers, the coefficient follows from the exact result. Once again, it is an open question whether a given model with given coefficients actually applies over a wide range of flows.

The original LRR model for the rapid pressure-strain term was linear in the Reynolds stresses and the mean velocity gradients, but, again, nonlinear models have been proposed more recently, partly to ensure realizability and proper limiting behavior at a solid surface (see the section). A significant recent development is the elliptic relaxation model of Durbin (1993), in which the pressure fluctuation and the pressure-strain term at a given point are affected by flow properties at all other points. Conventional models parameterize the pressure-strain term at a given point using only the velocity-field statistics at that point, which is clearly incorrect in view of the elliptic nature of the Poisson equation for the pressure.

The chief virtue of stress-transport models is that they do not use an eddy-viscosity relation between the Reynolds stresses and the components of the mean rates of strain. However, it is common to use a gradient-transport model for the turbulent transport terms. For example, $\partial\overline{u_i u_j u_l}/\partial x_l$ represents the turbulent transport of $\overline{u_i u_j}$ in the x_l direction, and $\overline{u_i u_j u_l}$ is assumed to be proportional to $-\partial\overline{u_i u_j}/\partial x_l$. The constant of proportionality is an eddy-diffusivity that could be taken as $c_\mu k^2/(\varepsilon\sigma)$—simply the eddy-viscosity from the k–ε model with an additional factor σ, which is adjusted to optimise the results. (In practice, a slightly more sophisticated and rotationally invariant model is used.) Obviously, the retention of eddy diffusivity is an embarrassment, and Huang et al. (1994) showed that particular difficulties occur in compressible flow where $\partial\bar{p}/\partial x_i$ appears, but alternative models for turbulent transport have not been very successful, and it is rare for errors in the gradient transport model to have a large effect on the flow.

A few closures at higher level than the Reynolds stresses have been tried, but without clear improvement. Two-point closures, using transport equations for the spatial covariance $\overline{u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})}$ between points at $\mathbf{x}$ and $(\mathbf{x}+\mathbf{r})$, become excessively complicated in inhomogeneous flows. A recent departure is the model of Reynolds and Kassinos (1995), using additional equations for turbulence structure (which is not adequately described by the Reynolds stresses and one-scalar length scale). It appears that stress-transport models, possibly with refinements such as those of Reynolds and Kassinos, will continue to be the most advanced type of Reynolds-averaged prediction method for the foreseeable future.

Stress-transport models are more difficult and expensive to run than eddy-viscosity models, because the indirectness of the coupling between the mean flow and the turbulence increases the stiffness of the system. Various workshops and other projects intended to find the most accurate type of model (e.g., Bradshaw et al. 1996) have not produced clear evidence in favor of stress-transport models. In many flows, they give significantly better results than do eddy-viscosity models, and, being closer to the physics, they are better vehicles for empirical allowances for special effects. However, performance in some complex flows is just as bad as eddy-viscosity models, and the likelihood of numerical singularity or near-singularity is greater. At present, therefore, most industrial companies stick to two-equation models, generally k–ε, despite their known shortcomings.

Conclusions

In 25 years, DNS and LES have become major players in turbulence research, and industrial use of LES for design purposes cannot be too far off. Also, there have been considerable advances in measurement techniques and in digital data acquisition and processing. On the other hand, the turbulence models in common use are rather modest refinements of those available in 1972. There are signs of real progress in modeling, but industry still awaits a model that is both reliable and cheap.

Acknowledgments

I am grateful to several colleagues, especially P. G. Huang of NASA Ames Research Center, D. M. McEligot of Idaho National Engineering Laboratory, and P. R. Spalart of The Boeing Company for discussions and for comments on a draft of this paper.

Appendix: Exact and model transport equations

The main equations all have the same form: the left-hand side is transport by the mean flow (substantial derivative), and the first term on the right is generation of the transported quantity by the influence of the mean flow. Succeeding lines show the pressure-strain term (individual Reynolds-stress transport equations only), the turbulent diffusion by velocity and pressure fluctuations, and the viscous destruction and diffusion. For simplicity, we consider only constant-property flow and use the symbol p, with fluctuation p', for (*pressure*)/(density).

If we multiply the x_j-component N–S equation by u_i, add it to u_j times the x_i-component equation, and take Reynolds average (usually a time average) we get the Reynolds-stress transport equation for $\overline{u_i u_j}$. Strictly speaking, the Reynolds-stress acting in the x_i-direction on a plane normal to the x_j-direction is $-\rho\overline{u_i u_j}$, but in the case of constant-property flow considered here, the difference is trivial. We have:

$$\begin{aligned} U_l\partial\overline{u_u u_j}/\partial x_l = &-(\overline{u_i u_l}\partial U_j/\partial x_l + \overline{u_j u_l}\partial U_i/\partial x_l) \\ &+\overline{p'(\partial u_i/\partial x_j + \partial u_j/\partial x_i)} \\ &-\partial\overline{u_i u_j u_l}/\partial x_l - \partial\overline{p' u_i}/\partial x_j - \partial\overline{p' u_j}/\partial x_i \\ &+\nu\left(\overline{u_i\partial^2 u_j/\partial u2_l} + \overline{u_j\partial^2 u_i/\partial x_l^2}\right) \end{aligned} \tag{A1}$$

A special case, obtained by putting $j = i$ and dividing by 2, is the transport equation for turbulent kinetic energy per unit mass, $\overline{u_i^2}/2$:

$$\begin{aligned} U_l\partial(\overline{u_i^2}/2)/\partial x_l = &-\overline{u_i u_l}\partial U_i/\partial x_l \\ &-\partial(\overline{p' u_l} + \overline{u_i^2 u_l}/2)/\partial x_l \\ &+\nu\overline{u_i\partial^2 u_i/\partial x_l^2} \end{aligned} \tag{A2}$$

The terms on the right can be rigorously interpreted as energy transfer from the mean flow to the turbulence (turbulent energy "production"), spatial transport of turbulent energy (known by the overworked word "diffusion"), and—splitting up the viscous term—energy transfer from the turbulence to thermal internal energy (viscous "dissipation"), and viscous transport of turbulent energy (again, negligible except in the viscous wall region or other situations where the local Reynolds number is low, so that the last term in Equation A2 can usually be replaced by $-\varepsilon$). Individual Reynolds-stress transport equations do not have this rigorous thermodynamic interpretation, but the general

processes of generation, turbulent transport, and destruction are the same. Note that the pressure-strain redistribution term $\overline{p'(\partial u_i/\partial x_j + \partial u_j/\partial x_i)}$ disappears from the TKE equation because $\partial u_i/\partial x_j = 0$ for $j = i$, by continuity.

The exact transport equation for the simplified "homogeneous" dissipation $\nu\overline{(\partial u_i/\partial x_j)^2}$, which is almost always an adequate approximation to the more complicated expression for the true dissipation, is still itself very complicated. The model equations bear so little resemblance to the exact equation that they should be thought of as purely empirical equations representing the usual processes of transport by the mean flow, generation by interaction with the mean flow, transport by the turbulence, and destruction by viscous stresses or pressure fluctuations.

The current version of the Jones–Launder k–ε model used by Launder's group at the University of Manchester, UK is:

Model equation for $k \equiv \overline{u_i^2}/2$

$$U_l\frac{\partial k}{\partial x_l} = -\overline{u_i u_l}\frac{\partial U_i}{\partial x_l} + \frac{\partial}{\partial x_l}\left[\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x_l}\right] - \varepsilon \tag{A3}$$

Model equation for ε

$$U_l\frac{\partial \varepsilon}{\partial x_l} = c_{\varepsilon_1}\frac{\varepsilon}{k}(-\overline{u_i u_l})\frac{\partial U_i}{\partial x_l} + \frac{\partial}{\partial x_l}\left[\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_l}\right] - c_{\varepsilon_2}\frac{\varepsilon^2}{k} \tag{A4}$$

Given k and ε, the eddy-viscosity follows from Equation 2. Numerical values are $c_\mu = 0.09$, $c_{\varepsilon_1} = 1.45$, $c_{\varepsilon_2} = 1.92$, $\sigma_k = 1.0$, $\sigma_\varepsilon = 1.3$. These values apply only at large local Reynolds number $\mathrm{Re}_T = k^2/(\varepsilon\nu)$.

The Launder-Reece-Rodi (1975) transport-equation model for high local Reynolds number $k^2/(\nu\varepsilon)$ (without later improvements) is:

Model equation for $\overline{u_i u_j}$

$$U_l\frac{\partial \overline{u_i u_j}}{\partial x_l} = -\left(\overline{u_i u_l}\frac{\partial U_j}{\partial x_l} + \overline{u_j u_l}\frac{\partial U_i}{\partial x_l}\right) + \Phi_{ij} + \frac{\partial}{\partial x_l}\left(c_s\overline{u_l u_m}\frac{k}{\varepsilon}\frac{\partial \overline{u_i u_j}}{\partial x_m}\right) - \frac{2}{3}\delta_{ij}\varepsilon \tag{A5}$$

where the pressure-strain "redistribution" term Φ_{ij} is modeled as $\Phi_{ij_1} + \Phi_{ij_2}$, with

$$\Phi_{ij_1} = -\frac{c_1\varepsilon}{k}\left[\overline{u_i u_j} - \frac{2}{3}\delta_{ij}k\right] \tag{A6}$$

$$\Phi_{ij_2} = c_2\left[\left(\overline{u_i u_l}\frac{\partial U_j}{\partial x_l} + \overline{u_j u_l}\frac{\partial U_i}{\partial x_l}\right) - \frac{1}{3}\delta_{ij}\overline{u_l u_m}\frac{\partial U_l}{\partial x_m}\right] \tag{A7}$$

The model equation for ε uses a diffusivity $c_\varepsilon\overline{u_l u_m}(k/\varepsilon)$ in the turbulent transport term; this is a slight improvement on the $(c_\mu/\sigma_\varepsilon)(k^2/\varepsilon)$ used in the k–ε method. The dissipation equation is otherwise identical to that of the k–ε method:

$$U_l\frac{\partial \varepsilon}{\partial x_l} = c_{\varepsilon_1}\frac{\varepsilon}{k}(-\overline{u_i u_l})\frac{\partial U_i}{\partial x_l} + \frac{\partial}{\partial x_l}\left(c_\varepsilon\overline{u_l u_m}\frac{k}{\varepsilon}\frac{\partial \varepsilon}{\partial x_m}\right) - c_{\varepsilon_2}\frac{\varepsilon^2}{k} \tag{A8}$$

Numerical values are $c_1 = 1.8$, $c_2 = 0.6$ (i.e., 3/5), $c_s = 0.22$, $c_{\varepsilon_1} = 1.45$, $c_{\varepsilon_2} = 1.92$, $c_\varepsilon = 0.18$. For a general 3-D flow, eleven PDEs—including the four mean-flow equations—must be solved, compared to six for the k–ε method.

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